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FAR-FIELD INTENSITY DISTRIBUTION
FROM A DIFFRACTING APERTURE
IN A TURBULENT MEDIUM

H. T. Yura

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PREFACE

This Memorandum, prepared for the Advanced Research Projects Agency, is part of a study of those phenomena which affect the performance of optical or infrared reconnaissance and guidance equipment. The objective of these studies is to provide sufficient understanding for the system analyst to compute performance estimates under various operational conditions.

A quantitative understanding of the effect of atmospheric turbulence on a beam of light of finite cross section is required for the prediction of the performance of various devices employing lasers for target acquisition or guidance in tactical missions. Such applications are characterized by near-horizontal propagation paths near the ground of the order of one to tens of kilometers in length. This Memorandum obtains the modified intensity distribution in the far field of a finite transmitting aperture due to a turbulent medium. These results should be of use to those interested in tactical applications of laser range finders, laser line scanners, and the various guidance systems employing an illuminating beam.

SUMMARY

The far-field intensity distribution at optical frequencies of an initially plane wave from a finite, circular, source aperture is obtained as a function of range and angle for various values of the index structure constant. Describing the turbulence-induced index of refraction fluctuations by the Kolmogorov spectrum, it is found that the time average of the peak radiant intensity in the Fraunhofer region of a finite source aperture decreases with range at a much faster rate than the intensity calculated from absorption and scattering by molecules and particulate matter. An expression is derived for this intensity as the sum of two terms: one which represents the exponential decay of the energy in the initially coherent beam; and the other which represents the complementary growth of the energy in an incoherent radiation field, attributed to the loss of coherence by scatterings off the turbulence-induced fluctuations in the medium. The exponent in this process is shown to be of the order $(R/R_c)^{11/6}$ (where R_c is the range where the average part of the field is down by a factor of the order e^{-1}) in contrast to the usual (R/R_v) for the attenuation by molecules and dust (where R_v is the visual range). The beamwidth at half-power is shown to increase very slowly until the propagation distance reaches a range that is of the order of R_c . These results can be illustrated by the application to a beam from an aperture of 2-cm diameter at a wavelength of 0.6328μ . For moderate daytime turbulence, $R_c \approx 1.7$ km at this wavelength, compared to an R_v of about 5 km for a correspondingly moderate visual range. For small values of l_0 (e.g., $l_0 \lesssim 0.5$ cm) where l_0 is the inner scale of turbulence, the effects of turbulence dominate over most of the range of interest. For this case, the beam will exhibit some structure out to a range of about $(1.0 - 1.65) R_c$ for $0.5 \text{ cm} \gtrsim l_0 \gtrsim 0.1 \text{ cm}$.

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I. INTRODUCTION

In the presence of turbulence, inhomogeneities appear in the medium which scatter electromagnetic waves. The electromagnetic field diffracted by an aperture exhibits, in the absence of turbulence, a characteristic diffraction pattern with well-defined regions of maximum and minimum intensity. The purpose of this Memorandum is to calculate, at optical frequencies, the effects of the turbulent medium on the diffraction pattern. It is shown that the beam pattern flattens and broadens as the propagation distance increases. Analytic expressions are derived for the intensity distribution of a plane wave incident on a circular aperture as a function of propagation distance and for the parameters that characterize the turbulent medium.

The results obtained here have application in predicting the performance of systems employing a laser for the detection and location of targets. The assumption is made that the parameters that characterize the turbulent medium are not a function of range. This is a good assumption for horizontal paths. However, these results may also be extended directly to apply to communications from satellites, etc.

The present analysis is restricted to a weakly inhomogeneous medium. That is, it is assumed that the fluctuations are small compared to the average properties of the medium. Furthermore, it is assumed that the characteristics of the medium do not change appreciably in a period of oscillation of the electromagnetic field, because frequency spreading (doppler effects) then becomes important. At near-infrared and optical wavelengths, this condition is satisfied in the atmosphere. The electromagnetic field under consideration has a time dependence given by the factor $e^{-i\omega t}$. In this case the time-dependent wave equation is replaced with the Helmholtz equation for an inhomogeneous medium. The electrical conductivity and magnetic permeability of the medium are taken to be zero and one respectively. Here, only the case of the propagation of a scalar field in a fluctuating medium is discussed. Extension to vector fields is straightforward. It is noted that the results obtained here apply not only to electromagnetic fields,

but to any field that satisfies the wave equation, e.g., acoustic waves when the scale of turbulence is greater than the wavelength. Furthermore, it is assumed that the diameter D of the aperture is larger than the inner scale of turbulence l_0 . Under this condition, effects such as beam steering may be neglected.

In Section II the physical nature of the electromagnetic field in a turbulent medium is reviewed. A previously demonstrated treatment of the field as a sum of an average and a random part is described. In Section III the time average of the electromagnetic field is discussed and applied to the field diffracted by an aperture. In Section IV the random field is discussed, and in Section V the intensity distribution as a function of range and field angle is obtained. It should be noted that these results are derived on the assumption of the single scattering approximation and therefore may not be accurate for $R \gg R_c$ (R_c is the range where the average part of the field is down by a factor of the order e^{-1}). Curves depicting the beam pattern (based on the Kolmogorov spectrum for the index of refraction fluctuations) as a function of various parameters are given. It is shown that the beam pattern flattens and broadens as the propagation distance R increases. In particular, when $R > R_M$ (defined below) the beam pattern is essentially flat.

The analytic results obtained are valid for propagation distances $R > R_1 = D^2/\lambda$, where D and λ are the diameter of the aperture and wavelength of the field respectively; i.e., the results are valid in the Fraunhofer region. It is assumed here that $R_1 < R_c$. In this case, the modification of the near-field diffraction pattern (Fresnel region) is entirely negligible.

II. GENERAL CONSIDERATIONS

In this section the qualitative nature of an electromagnetic wave propagating in a weakly inhomogeneous medium is reviewed briefly.⁽¹⁾ U , a typical component of the field, can be represented as the sum of two terms: $U_{\text{avg}}(\vec{R})$, the average field defined by $U_{\text{avg}} = \bar{U}$; and a random field U_{ran} where $\bar{U}_{\text{ran}} = 0$.^{*} As the field progresses through the medium, energy is transferred from the average part to the random part of the field so that energy is conserved. For propagation distances R much less than a critical distance R_c (defined below) the field is primarily unscattered; the scattered component is of the order n_1 where n_1 is the fluctuating part of the index of refraction (i.e., $n = 1 + n_1$, $|n_1| \ll 1$, $\bar{n}_1 = 0$). For propagation distances much greater than R_c the field is essentially random since the average field is exponentially small (for $R \gg R_c$ the field has undergone multiple scatterings off the random inhomogeneities of the medium and is completely random).

For $R \ll R_c$, the total field

$$U(\vec{R}) = U_{\text{avg}}(\vec{R}) + U_{\text{ran}}(\vec{R}) \quad (1)$$

where, to terms up to second order in n_1 , $U_{\text{avg}}(\vec{R})$ is formally the same as the field in the absence of turbulence (i.e., for $n_1 \equiv 0$), except that the propagation wave vector k is not given by its value in the absence of turbulence (k_0) but is given by⁽¹⁾

$$k = k_0 \left\{ 1 + \frac{B_n(0)}{2} + \frac{k_0^2}{2\pi} \phi_n(k_0) \right\} \quad (2)$$

where $B_n(0)$ is the correlation function of the index of refraction fluctuation evaluated at $r = 0$,^{**} i.e.,

^{*} A bar over a quantity signifies the ensemble average of the quantity.

^{**} It is assumed here that the medium is statistically homogeneous and isotropic.

$$\begin{aligned}
 B_n(\vec{r}) &= \overline{n_1(\vec{r}_1)n_1(\vec{r}_2)} \\
 &= B_n(\vec{r}_1 - \vec{r}_2) \\
 &= B_n(r) \quad \text{where} \quad r = |\vec{r}_1 - \vec{r}_2|
 \end{aligned}$$

and

$$\phi_n(k_o) = \int d\vec{r} \frac{e^{ik_o r}}{r} B_n(\vec{r}) e^{i\vec{k}_o \cdot \vec{r}} \quad (3)$$

It can be seen that $\phi_n(k_o)$ (and hence k) has a positive imaginary part. ⁽¹⁾

The random part of the field, to lowest order in n_1 , is given by the first Born approximation:

$$U_{\text{ran}}(\vec{R}) = U_o(\vec{R})\psi_1(\vec{R}) \quad (4)$$

where $U_o(\vec{R})$ is the field in the absence of turbulence, and $\psi_1(\vec{R})$ is given by ⁽²⁾

$$\psi_1(\vec{R}) = \frac{k_o^2}{2\pi U_o(\vec{R})} \int n_1(\vec{R}') U_o(\vec{R}') \frac{e^{ik_o |\vec{R} - \vec{R}'|}}{|\vec{R} - \vec{R}'|} d\vec{R}' \quad (5)$$

where the integration volume is that region of space where $n_1(\vec{R}) \neq 0$ (the aperture being located at $R = 0$).

III. THE AVERAGE FIELD

Consider an expanding spherical wave. In the absence of turbulence this field is given by

$$U_o(R) = C \frac{e^{ik_o R}}{R} \quad (6)$$

where C is an arbitrary constant. For $R \ll R_c$ the average part of the spherical wave is given by

$$\begin{aligned} U_{avg}(R) &= C \frac{e^{ikR}}{R} \\ &= C \frac{e^{ik_o R}}{R} \left(e^{i(k-k_o)R} \right) \end{aligned} \quad (7)$$

Let

$$\phi = i(k - k_o)R \quad (8)$$

Then

$$U_{avg}(R) = \frac{Ce^{ik_o R}}{R} \left[e^{Re\phi} + iIm\phi \right] \quad (9)$$

To lowest order in n_1

$$Re\phi = -\frac{k_o^3 R}{2\pi} Im\epsilon_n(k_o) \quad (10)$$

and

$$\text{Im}\phi = \left[\frac{B_n(0)}{2} + \frac{k_o^2}{2} \text{Im}\phi_n(k_o) \right] k_o R \quad (11)$$

Thus, in the presence of the random inhomogeneities, the average component of the field possesses a positive attenuation coefficient. The quantitative results given by Eqs. (2) and (5) are obtained by a perturbation method valid for $R < R_c$. The critical distance R_c is obtained from the requirement that the relative magnitude of the correction to the field in the absence of turbulence be small. It can be seen that this requirement leads to the condition that⁽¹⁾

$$|\text{Re}\phi| \ll 1 \quad (12)$$

The critical distance R_c is determined by

$$|\text{Re}\phi(R_c)| = 1 \quad (13)$$

Thus, R_c is the propagation distance where the average component of the field is down by a factor of the order e^{-1} .

It follows immediately from the optical theorem⁽¹⁾ (or from energy conservation) that

$$\begin{aligned} -2\text{Re}\phi(R) &= \overline{|\psi_1(R)|^2} \\ &= \overline{[\text{Re}\psi_1(R)]^2} + \overline{[\text{Im}\psi_1(R)]^2} \end{aligned} \quad (14)$$

where ψ_1 is given by Eq. (5). Hence R_c may also be determined from

$$1/2 \overline{|\psi_1(R_c)|^2} = 1 \quad (15)$$

Furthermore, if $(\lambda_o R)^{1/2} \gg \ell_o$, where ℓ_o is the inner scale of turbulence,^{*} it can be shown that⁽²⁾

$$\overline{(\text{Re}\psi_1)^2} \sim \overline{(\text{Im}\psi_1)^2} \quad (16)$$

In this case R_c may be determined from

$$[\overline{\text{Re}\psi_1(R_c)}]^2 = 1, \quad (\lambda_o R)^{1/2} \gg \ell_o \quad (17)$$

Choosing the Kolmogorov spectrum to represent the index of refraction fluctuations, it can be shown that for a spherical wave and $(\lambda_o R)^{1/2} \gg \ell_o$

$$\overline{(\text{Re}\psi_1)^2} \cong 0.13 C_{n_o}^2 k_o^{7/6} R^{11/6} \quad (18)^{**}$$

where C_n^2 is the index structure function.⁽²⁾ Strictly speaking, Eq. (18) is valid for propagation paths where C_n^2 is independent of position along the path, e.g., horizontal paths. For slant paths through the atmosphere, Eq. (18) (for spherical waves) is replaced by

$$\overline{(\text{Re}\psi_1)^2} = 0.238 k_o^{7/6} \int_0^R C_n^2(h)_z^{5/6} dz \quad (18a)$$

where $h = z \cos \alpha$, and α is the zenith angle.^{***} When C_n^2 is a constant, Eq. (18a) is identical to Eq. (18). In this paper it is assumed that C_n^2 is a constant. Extension to the more general case is straightforward.

^{*}Typically, $0.1 \text{ cm} \leq \ell_o \leq 1 \text{ cm}$ for horizontal paths near sea level.

^{**}See Ref. 2, Eq. 9.43.

^{***}See Ref. 2, Chap. 8.

The values of C_n^2 typical of turbulence in the first few hundred meters of the atmosphere are: $30 \times 10^{-15} \text{ cm}^{-2/3}$ (strong daytime turbulence within a few meters of the ground), $3 \times 10^{-15} \text{ cm}^{-2/3}$ and $7 \times 10^{-15} \text{ cm}^{-2/3}$ (moderate daytime turbulence and/or strong nighttime turbulence), and $0.3 \times 10^{-15} \text{ cm}^{-2/3}$ (very weak turbulence which occurs in the near neutral periods at dawn or dusk). Figure 1 gives R_c as a function of wavelength for various values of C_n^2 . For example, for $\lambda_0 = 0.6328 \mu$ (He-Ne laser wavelength) it is found that

$$\left. \begin{aligned} R_c &\cong 5.9 \text{ km}, & C_n^2 &= 0.3 \times 10^{-15} \text{ cm}^{-2/3} \\ &\cong 3.1 \text{ km}, & C_n^2 &= 1 \times 10^{-15} \text{ cm}^{-2/3} \\ &\cong 1.7 \text{ km}, & C_n^2 &= 3 \times 10^{-15} \text{ cm}^{-2/3} \\ &\cong 1.1 \text{ km}, & C_n^2 &= 7 \times 10^{-15} \text{ cm}^{-2/3} \\ &\cong 0.48 \text{ km}, & C_n^2 &= 30 \times 10^{-15} \text{ cm}^{-2/3} \end{aligned} \right\} \quad (19)$$

For plane waves, R_c is equal to 0.61 times the values for a spherical wave given by Eq. (19).

Next, consider the average field diffracted by an aperture. In the Fraunhofer region, the field in the absence of turbulence is given by ⁽³⁾ (it is assumed here that $\lambda_0/D \ll 1$ where D is a characteristic length of the aperture)

$$U_o(\vec{R}) = \frac{k_o u_o}{2\pi i} \left(\frac{e^{ik_o R}}{R} \right) \int e^{i(\vec{k}_o - \vec{k}_o') \cdot \vec{\rho}} dA_n \quad (20)$$

where u_o is the magnitude of the field (assumed constant) in the aperture, $\vec{\rho}$ is the vector from the origin to an arbitrary point in the

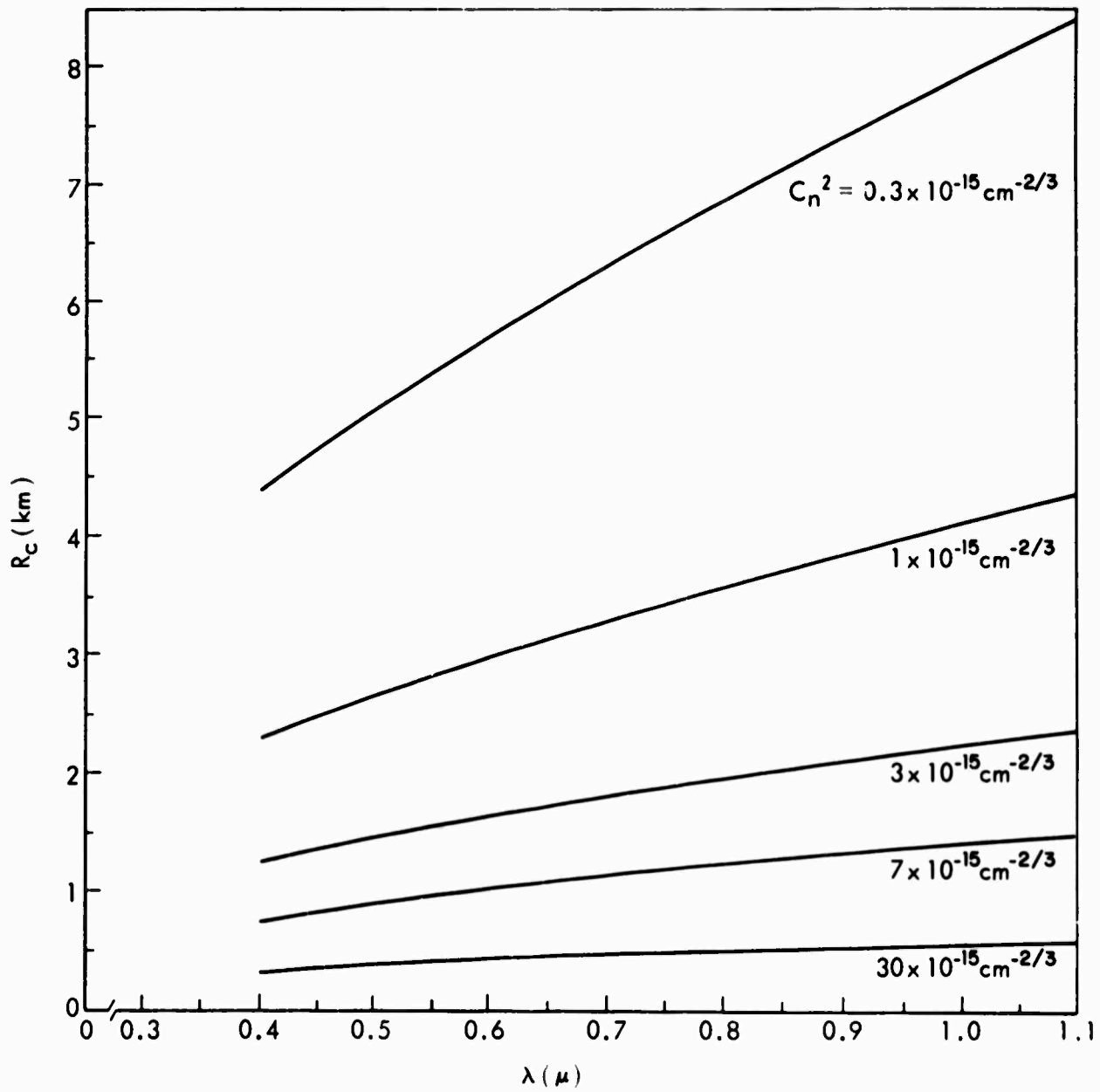


Fig. 1—The critical distance R_c as a function of wavelength

aperture, \vec{k}'_0 is the wave vector of the diffracted wave (differing in direction only from \vec{k}_0 which is assumed nearly normal to the aperture), and dA_n is the projection of an area element of the aperture normal to the direction of the initial (undiffracted) light wave.

In the presence of the random medium the average field is given by Eq. (20), with k_0 replaced by k where k is given by Eq. (2). For a circular aperture of radius a , from Eq. (20)

$$U_{\text{avg}}(R, \theta) = \frac{u_0 \epsilon}{1} \left(\frac{e^{ikR}}{R} \right) \left(\frac{J_1(a\chi)}{\chi} \right) \quad (21)$$

where

$$\left. \begin{aligned} \epsilon &= ka \\ \chi &= k\theta \\ \sin \theta &\cong \theta \end{aligned} \right\} \quad (22)$$

and k is given by Eq. (2).

To lowest order in n_1

$$\begin{aligned} U_{\text{avg}}(R, \theta) &= \frac{u_0 \epsilon}{1} \left(\frac{e^{ikR}}{R} \right) \left(\frac{J_1(a\chi)}{\chi} \right) \\ &\cong \left\{ \frac{u_0 \epsilon}{1} \left(\frac{e^{ik_0 R}}{R} \right) \left(\frac{J_1(a\chi_0)}{\chi_0} \right) \right\} \exp \left[-\frac{1}{2} |\psi_1(\vec{R})|^2 \right] + O(n_1^4) + \dots \quad (23) \end{aligned}$$

where $\chi_0 = k_0 \theta$ and $\psi_1(\vec{R})$ is given by Eq. (5). The terms in the braces in Eq. (23) represent the diffracted field in the absence of turbulence, and the effects of the random medium are represented by the exponential term. The analytic expression for ψ_1 given by Eq. (5) is valid for $R < R_c$. On physical grounds it is expected that the qualitative nature of the average field remains the same for $R \gtrsim R_c$; i.e., it is damped. In particular, for large propagation distances, $U_{\text{avg}} = 0$, all of the field energy having been transferred to the random field.

IV. THE RANDOM FIELD

The random field for $R \ll R_c$ is given by Eq. (5). In this expression the quantity $U_0(\vec{R})$ is the field that is present in the absence of turbulence. In the case of Fraunhofer diffraction by an aperture (located at the origin of coordinates), the random field ψ_1 is thus given by Eq. (5) with U_0 given by Eq. (20). In the integral of Eq. (5) it is seen that an integration of $U_0(R')$ over all values of \vec{R}' is required. Equation (20) gives the diffracted field in the Fraunhofer region ($R > D^2/\lambda$). For $R \ll D^2/\lambda$, the Fresnel expression for the diffracted field should be used. In this report interest is in large propagation distances ($R \gg D^2/\lambda$); hence negligible error is introduced in the volume integration of Eq. (5) by using Eq. (20) for all values of R .

The integral obtained by substituting Eq. (20) into Eq. (5) presents great mathematical difficulty. However, if we consider the case

$$k_0 a / KR \ll 1 \quad (24)$$

(where K is the wave number of the power spectrum of the index of refraction fluctuation) it can be shown that the resulting integral expression obtained reduces to that given by Tatarski for a spherical wave.* Furthermore, long propagation distances where $(\lambda_0 R)^{1/2} \gg l_0$ are of interest. In this case the spectrum of the correlation function for the incoherent field fluctuation is concentrated near

$$K \simeq 2\pi / (\lambda_0 R)^{1/2} \quad (25)$$

(i.e., the refractive index inhomogeneities with scales of the order $(\lambda_0 R)^{1/2}$ make the largest contribution to the field fluctuations of the wave).** From Eqs. (24) and (25) we find that in the case of

* See Ref. 2, Chap. 9.

** See Ref. 2, Chap. 7.

$$R \gg R_1 = \frac{D^2}{\lambda_0} \quad (26)$$

Tatarski's results for a spherical wave may be used for the integral expression of $\psi_1(\vec{R})^2$. For a wide range of parameters, $R_1 \ll R_c$. For $\lambda_0 = 0.6328\mu$ and $a = 1$ cm, $R_1 \cong 600$ m, while $R_c > 1$ km (see Eq. (19)).

Results derived here apply to the case $R > R_1 = D^2/\lambda_0$. For $R < R_1$ a more thorough analysis of the integral expression for ψ_1 (i.e., Eq. (5)) with U_0 given by Eq. (20) is needed. No qualitative calculations for $R < R_1$ are attempted here.

V. BEAM PATTERN

In this section the results of Sections III and IV are used to obtain the diffraction pattern of a circular aperture in the presence of the inhomogeneous medium. First, an approximate expression for the diffracted power valid for $R \sim R_c$ is derived. To do this it is assumed that the average radiated power can be written as a sum of two terms for all values of R : an average part (proportional to $|U_{\text{avg}}|^2$), and a random part (proportional to $|U_{\text{ran}}|^2$). This, in effect, assumes that the phase of the random part of the field is not correlated to the phase of U_{avg} . This is true both for $R \gg R_c$ and for $R \ll R_c$; i.e., $\overline{U_{\text{avg}} U_{\text{ran}}} = 0$ for both of these limiting cases, and it is also assumed to be true for $R \sim R_c$. In order to obtain the angular dependence of the beam pattern it is noted that the angle of scattering of the field by the refractive index inhomogeneities is of the order λ/ℓ_0 , where ℓ_0 is the inner scale of turbulence (see Ref. 2). The assumption that the scattering angle is of the order λ/ℓ_0 is consistent with the observation that stellar images even under poor seeing conditions exhibit a finite blur circle.⁽⁴⁾ In the absence of turbulence almost all of the diffracted power is contained in a cone of the order λ/D . Since $\ell_0 < D$ (typically, $0.1 \text{ cm} \leq \ell_0 \leq 1 \text{ cm}$), it is seen that most of the scattered power will be contained in a cone with angular aperture of the order λ/ℓ_0 . Hence, to obtain the beam pattern we assume that the scattered radiation is contained in this cone. This neglects the relatively small amount of radiation contained in the sidelobes. In view of other approximations involved in the calculation and uncertainties in the underlying statistical description of the index of refraction fluctuations, it is felt that this approximation is justifiable.

Therefore, let ($d\Omega = 2\pi \sin \theta d\theta$)

$$\overline{dP} = \overline{dP}_{\text{avg}} + \overline{dP}_{\text{ran}} \quad (27)$$

where

$$\overline{dP}_{\text{avg}} = \frac{P_0 \epsilon^2}{4\pi} \Lambda_1^2(\chi a) e^{-f(R)} d\Omega \quad (28)$$

P_0 is the power in the aperture, and $\Lambda_1(x) [= 2J_1(x)/x]$ is the Lommel function of the first kind⁽⁵⁾ whose square gives the beam pattern in the absence of turbulence, and

$$\overline{dP}_{\text{ran}} = \frac{C}{2\pi} (1 - e^{-f(R)}) d\Omega \quad (29)$$

The constant C in Eq. (29) is determined by requiring that $\int dP = P_0$ (the power in the aperture), which yields that $C = 2P_0 \left(\frac{\ell_0}{\lambda}\right)^2$. The quantity βR is determined by the requirement that for small R , the power given by Eq. (27) gives the perturbation result. Doing this we obtain

$$\begin{aligned} f(R) &= 2[\overline{\text{Re}\psi_1(R)}]^2 \\ &= 0.26 C_n^2 k^{7/6} R^{11/6} \quad (\text{for the Kolmogorov spectrum}) \end{aligned} \quad (30)$$

Hence, from Eqs. (27)-(30)

$$\overline{dP} = \frac{P_0 \epsilon^2}{4\pi} \left\{ \Lambda_1^2(\epsilon\theta) e^{-\overline{|\psi_1|^2}} + \frac{2(1 - e^{-\overline{|\psi_1|^2}})}{\gamma^2} \right\} d\Omega \quad (31)$$

where

$$\gamma^2 = \frac{\pi(D)}{8(\ell_0)}^2 > 1$$

For $R \gg R_c$, $dP = P_0 d\Omega$; the average field being exponentially small, the quantity in the braces in Eq. (31) represents the modified beam pattern due to the effects of turbulence. Denoting this quantity by $\Lambda_1^2(\epsilon\theta) F(R, \theta)$, we have

$$\Lambda_1^2(\epsilon\theta) F(R, \theta) = \Lambda_1^2(\epsilon\theta) \left[e^{-\overline{|\psi_1(R)|^2}} + \frac{2(1 - e^{-\overline{|\psi_1(R)|^2}})}{\gamma^2 \Lambda_1^2(\epsilon\theta)} \right] \quad (32)$$

where, to lowest order in $\overline{n_1^2}$, $\epsilon = k_0 a = 2\pi a/\lambda \ll 1$. The quantity in the brackets in Eq. (32) thus represents the effects of the turbulent medium on the beam pattern.

A quantity of interest is the dependence of the radiant intensity on range for $\theta = 0$. Figures 2 and 3 are a plot of $\Lambda_1^2(0) F(R,0)$ as a function of propagation distance R for the Kolmogorov spectrum for various values of the index structure constant, ℓ_0 and $\lambda = 0.6328\mu$ and 1.06μ .^{*} In addition to turbulent scattering, there is an exponential loss of intensity due to both absorption and scattering by molecular and particulate constituents of the atmosphere. This attenuation is given by $\exp[-R/R_V]$ where R_V is the visual range.

Commonly occurring values of R_V are 1-5 km. Included in Figs. 2 and 3 is a plot of this attenuation for $R_V = 1$ and 5 km. It is seen that even with moderate turbulence and $D = 2$ cm, the attenuation of the beam will be dominated by turbulent scattering when $\ell_0 \lesssim 0.5$ cm. For larger values of D within the range of interest it is expected that turbulent scattering will be dominant for $\ell_0 \lesssim 1$ cm. In Figs. 4 and 5 the pattern (i.e., Eq. (32)) is plotted as a function of $t (= \epsilon\theta)$ for various values of propagation distance R , ℓ_0 , and $C_{no}^2 = 3 \times 10^{-15} \text{ cm}^{-2/3}$ for a 2-cm diameter aperture. The quantity $\Lambda_1^2(t)$ gives the beam pattern in the absence of turbulence. The beam pattern for other values of C_n^2 (for fixed ℓ_0 , λ , and D) may be obtained from Figs. 4 and 5 by replacing R by $R[C_{no}^2/C_n^2]^{6/11}$ where $C_{no}^2 = 3 \times 10^{-15} \text{ cm}^{-2/3}$. The sharp discontinuities shown in Figs. 4c and 5c result from the assumption that the scattered power is contained in a cone of half-angle equal to λ/ℓ_0 and are not physically significant.

Another quantity of interest is the half-power width. That is, that value of θ where the radiant intensity is one-half its value at $\theta = 0$. From Eq. (32) the angle $\theta_{1/2}$ at half-maximum is given by the solution to

^{*} Note that $\Lambda_1(0) = 1$; in the absence of turbulence the normalized Fraunhofer beam pattern is independent of range.

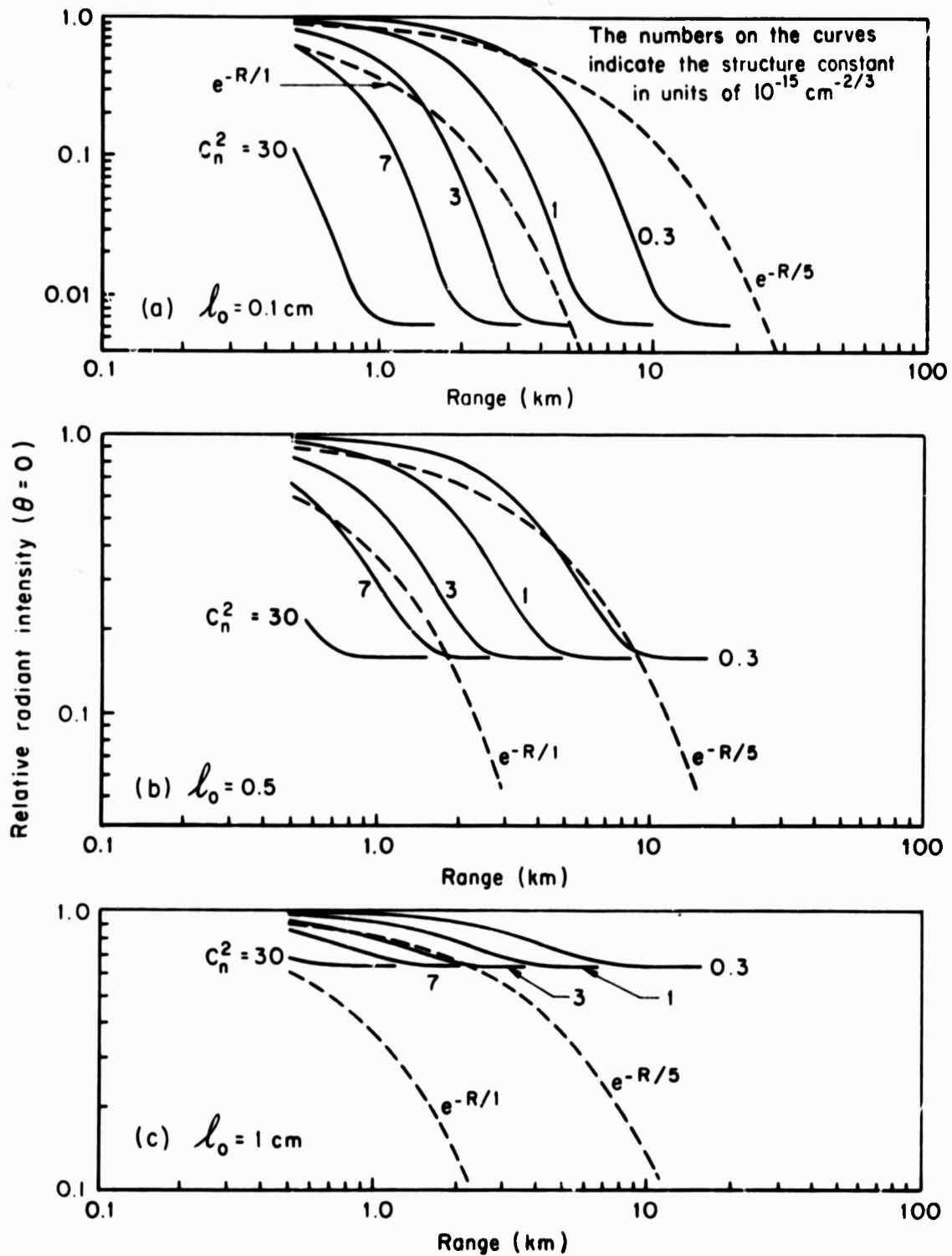


Fig. 2—The dependence of radiant intensity on range for $\theta = 0$ ($D = 2$ cm, $\lambda = 0.6328 \mu$) and various values of l_0

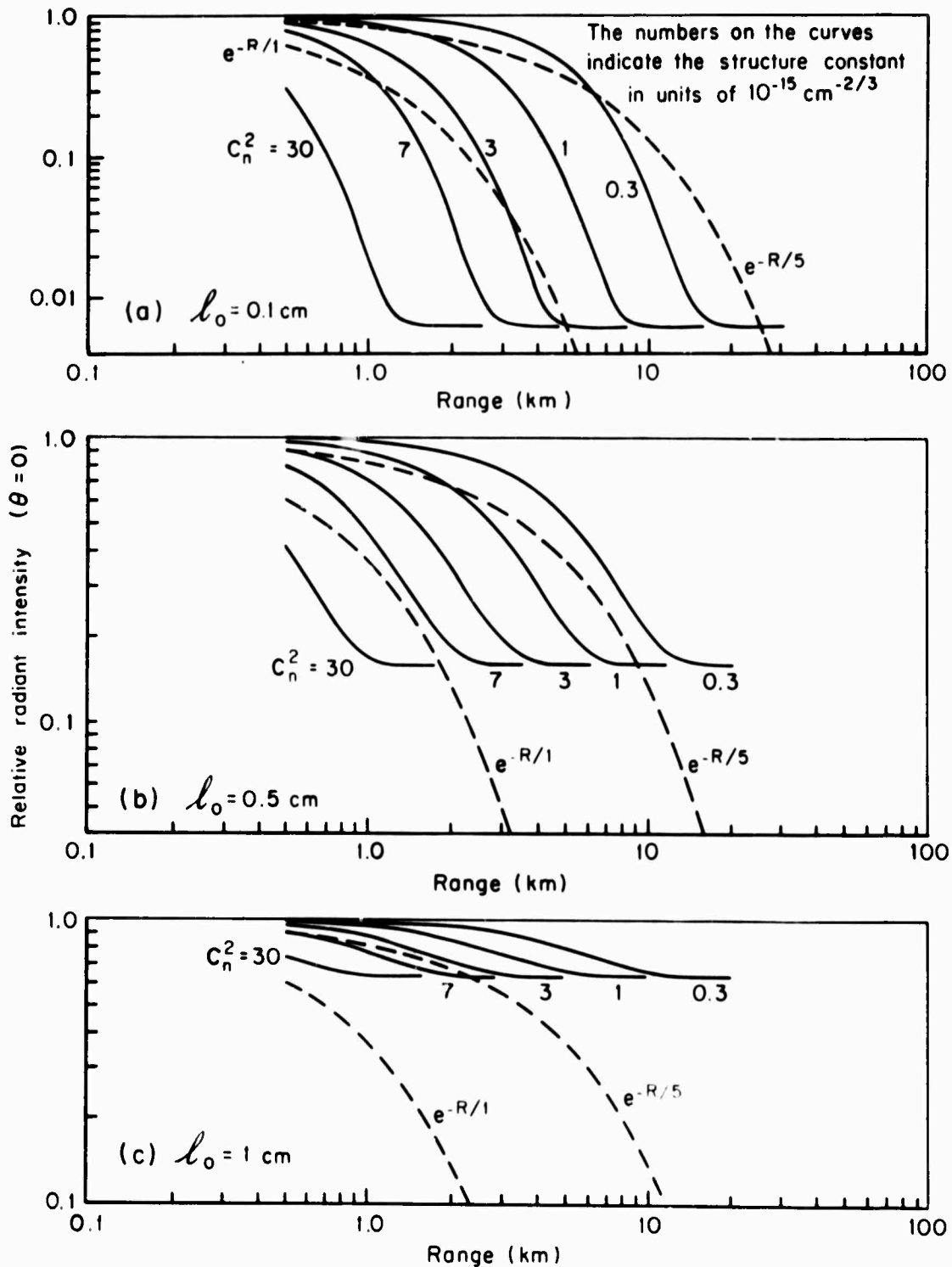


Fig. 3—The dependence of radiant intensity on range for $\theta = 0$ ($D = 2$ cm, $\lambda = 1.06 \mu$) and various values of l_0

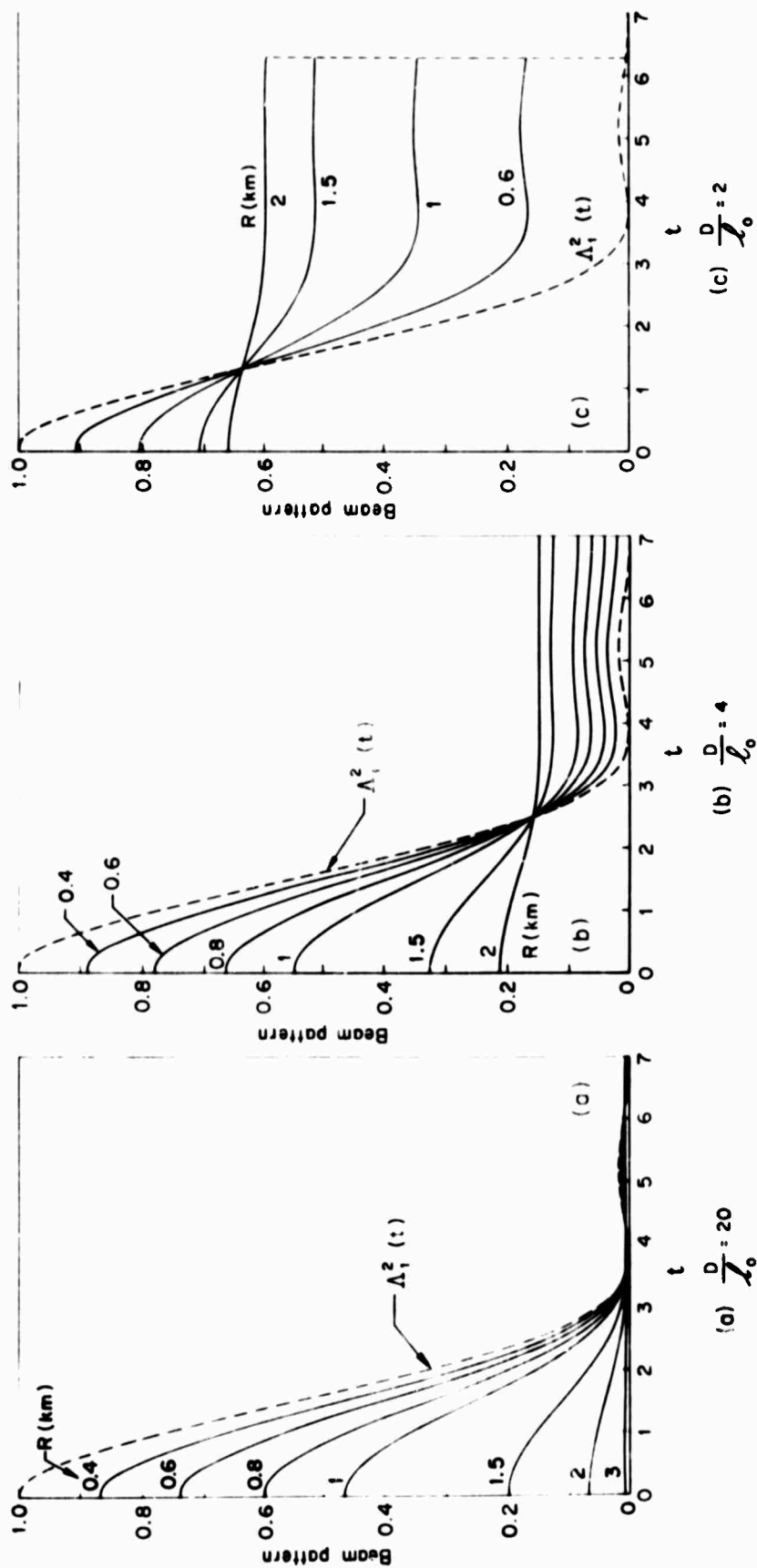


Fig. 4—Beam pattern as a function of t ($=\epsilon\theta$) for various values of l_0 and R
 $(\lambda = 0.6328\mu, R_c = 1.7 \text{ km, and } C_{n0}^2 = 3 \times 10^{-15} \text{ cm}^{-2/3})$

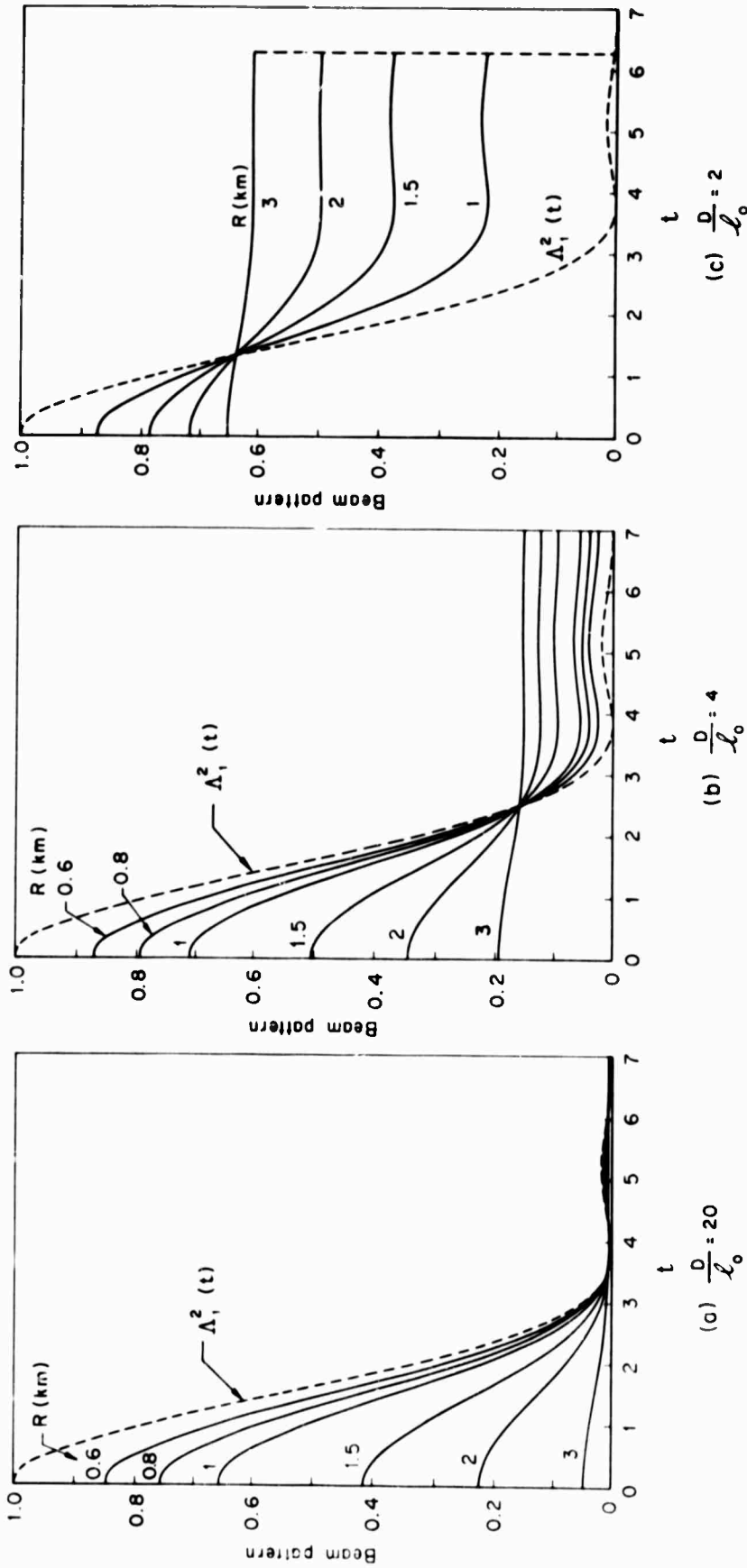


Fig. 5—Beam pattern as a function of t ($=\epsilon\theta$) for various values of L_0 and R
 $(\lambda = 1.06\mu, R_c = 2.3 \text{ km, and } C_{n0}^2 = 3 \times 10^{-15} \text{ cm}^{-2/3})$

$$\left(\frac{2J_1(t_{1/2})}{t_{1/2}}\right)^2 e^{-f(R)} + \frac{2}{\gamma^2} \left(1 - e^{-f(R)}\right) = \frac{1}{2} \left[e^{-f(R)} + \frac{2}{\gamma^2} \left(1 - e^{-f(R)}\right) \right] \quad (33)$$

where

$$t_{1/2} = \epsilon \theta_{1/2} = \frac{2\pi a}{\lambda} \theta_{1/2} \quad (34)$$

Rearranging terms in Eq. (33),

$$\left(\frac{2J_1(t_{1/2})}{t_{1/2}}\right)^2 = \frac{1}{2} \left[1 - \frac{2}{\gamma^2} (e^{f(R)} - 1) \right] \quad (35)$$

Equation (35) has been solved numerically for $t_{1/2}$ as a function of range and is plotted in Figs. 6 and 7 for various values of C_n^2 , ℓ_0 , and λ . As R increases, $t_{1/2}$ increases slowly until the propagation distance reaches a range that is of the order of R_c , where it increases rapidly to a limiting value equal to $\pi D/\ell_0$. The curves in Figs. 6 and 7 are for range values such that $t_{1/2} < \pi D/\ell_0$, the quantity $t_0 (\cong 1.62)$ is the half-power width in the absence of turbulence.

The diffracted beam flattens and broadens as R increases. That is, for large enough R , the beam pattern is contained in a cone angle of the order λ/ℓ_0 . For values of $t (= \epsilon \theta)$ near the central maximum of the beam pattern ($\sim \pi$), R_M is defined as that range such that for $R > R_M$ the radiant intensity never varies by more than one-half its maximum value. For $R > R_M$ the effect of the transmitting aperture is lost. We note that $t_{1/2} \gg 1$ at this range; hence, from Eq. (35)

$$1 - \frac{2}{\gamma^2} \left(e^{f(R_M)} - 1 \right) \cong 0$$

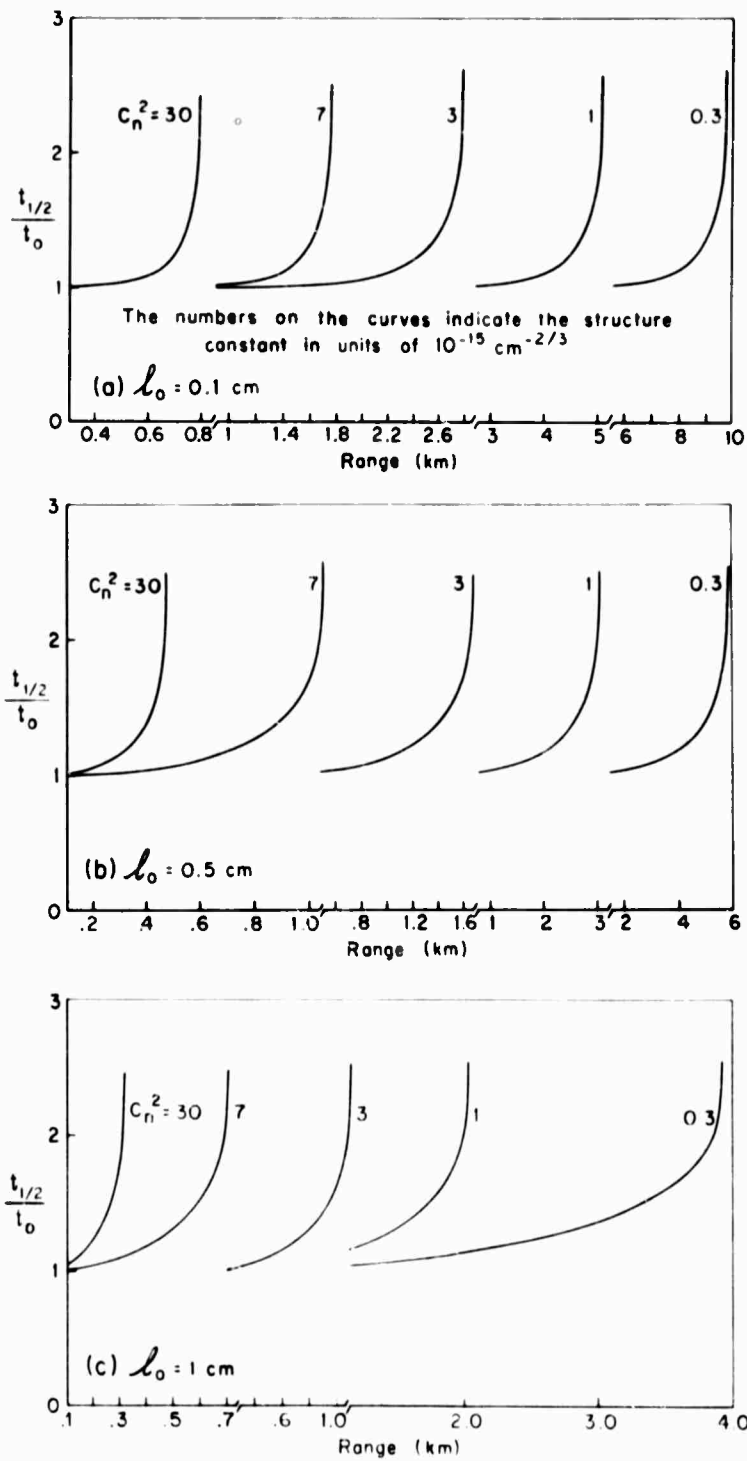


Fig. 6—The normalized half-power width $t_{1/2}/t_0$ as a function of range for $\lambda = 0.6328\mu$, $D = 2 \text{ cm}$, various values of the index structure constant, and l_0

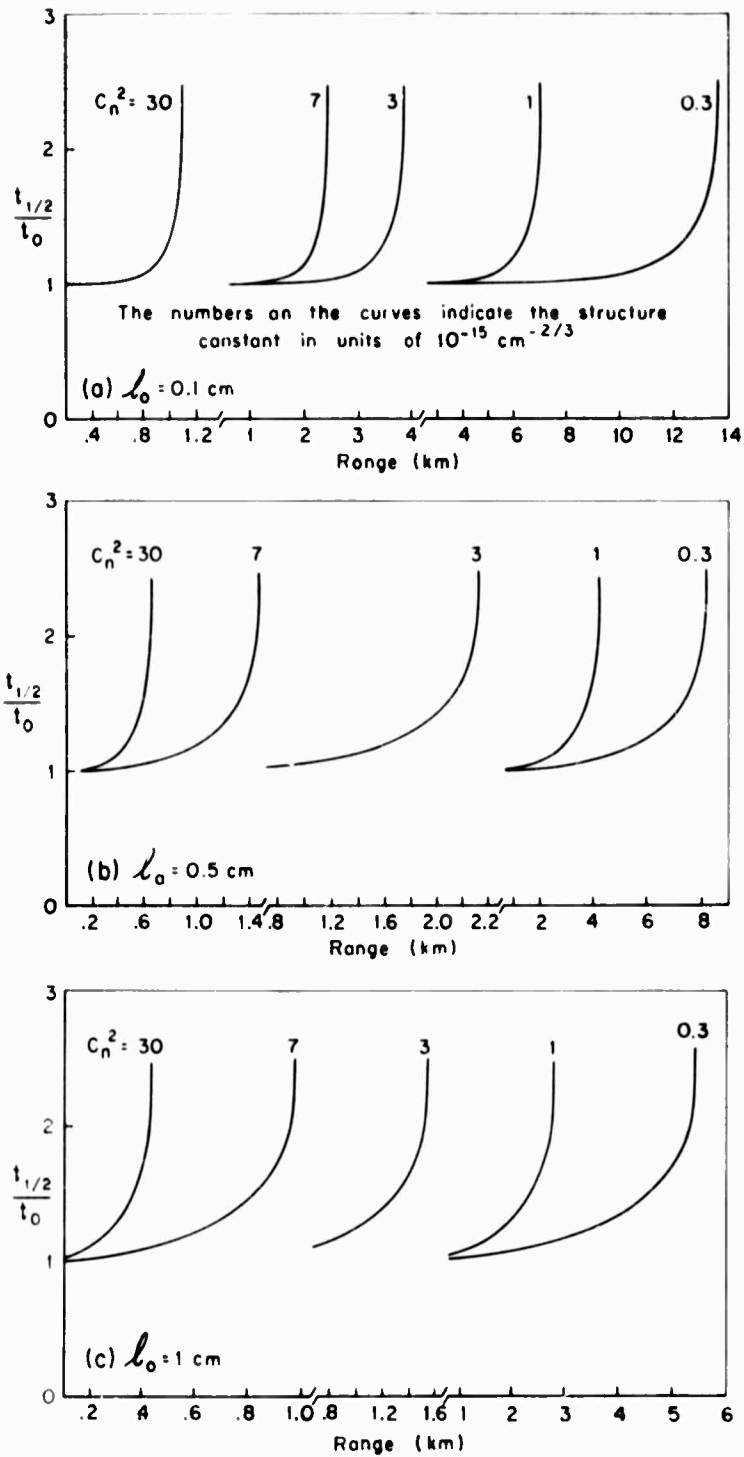


Fig. 7—The normalized half-power width $t_{1/2}/t_0$ as a function of range for $\lambda = 1.06 \mu$, $D = 2 \text{ cm}$, various values of the index structure constant, and l_0

or

$$f(R_M) = \log \left(\frac{\gamma^2}{2} + 1 \right) \quad (36)$$

From Eqs. (30) and (36), we obtain

$$R_M = \frac{\left[\log \left(\frac{\gamma^2}{2} + 1 \right) \right]^{6/11}}{0.26 C_n^2 k^{7/6}} \quad (37)$$

$$= \left[\frac{1}{2} \log \left(\frac{\gamma^2}{2} + 1 \right) \right]^{6/11} R_c$$

Thus, as R increases, $t_{1/2}$ increases from its value in the absence of turbulence ($t_0 = 1.62$), tending rapidly to its limiting value ($\pi D / l_0$) as $R \rightarrow R_M$. Figure 8 is a plot of R_M , for a 2-cm diameter aperture, as a function of wavelength for various values of C_n^2 . Figure 9 is a plot of the coefficient of R_c in Eq. (37). This factor is independent of wavelength and turbulence strength and is only a function of l_0 and D .

These results have been derived assuming the single scattering approximation. It is expected that this assumption is valid for $R \lesssim R_c$. Therefore the asymptotic behavior (for $R > R_c$) shown in Figs. 2 and 3 may be an artifact of this approximation.

These results can be illustrated by the application to a beam from an aperture of 2-cm diameter at a wavelength of 0.6328 μ . For moderate daytime turbulence, $R_c \approx 1.7$ km at this wavelength compared to an R_v of about 5 km for a correspondingly moderate visual range. However, for $l_0 \lesssim 0.5$ cm, the difference in functional dependence results in a crossover range with the effects of turbulence dominant over most of the range of interest. For this case, the beam will exhibit some structure out to a range of about $(1.0 - 1.65) R_c$ for $0.5 \text{ cm} \gtrsim l_0 \gtrsim 0.1 \text{ cm}$.

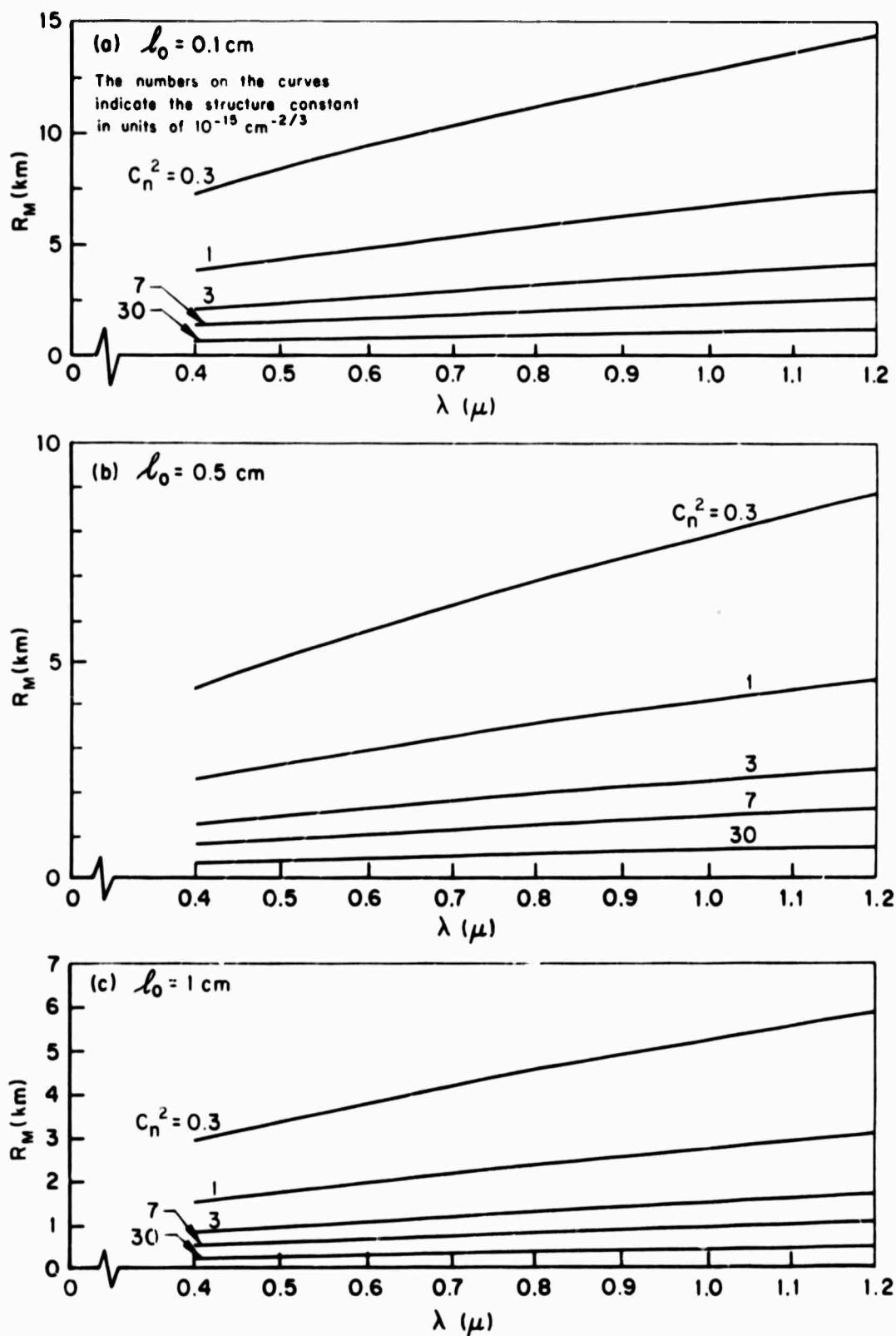


Fig. 8—The maximum range R_M as a function of wavelength for various values of the index structure constant, $D = 2$ cm, and l_0

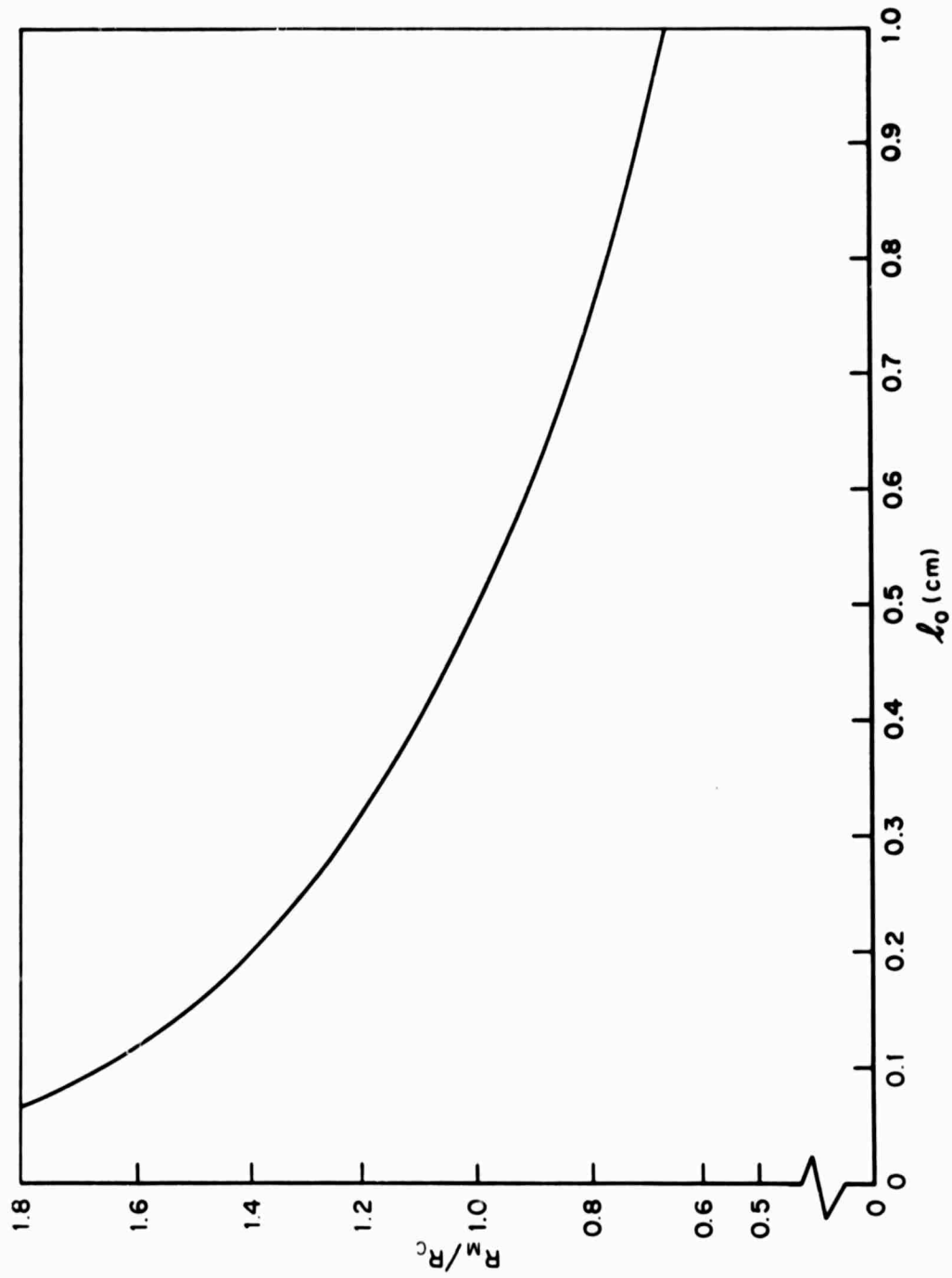


Fig. 9—The ratio R_M/R_C as a function of l_0 for a 2-cm diameter beam aperture

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